THE 3D BOUSSINESQ SYSTEM WITH PARTIAL VISCOSITY AND LARGE DATA IN THE THIN DOMAIN

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Abstract. In this paper, we study the initial boundary value problem of Boussinesq equations with partial viscosity on a three-dimensional (3D) thin domain. The global well-posedness of strong solution with initial data $(u_0, \theta_0) \in H^2(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$ and suitable boundary conditions is established, where $\Omega_\varepsilon = \Omega \times [0, \varepsilon] \subset R^3$ and $\Omega \subset R^2$ are both bounded domains with smooth boundaries.

Subsequently, when $\varepsilon \to 0$, we study the asymptotic behavior of the strong solution to the 3D thin domain system.

Keywords: 3D thin domain, Incompressible fluid, Large data, Partial viscosity.

Mathematics subject classication: 35Q35, 35Q80, 76N10.

1. Introduction

This paper is devoted to study the following incompressible Boussinesq system:

$$
\begin{cases}
\partial_t u + (u \cdot \nabla)u - \nu_1 \Delta u + \nabla P = \theta e_3, \\
\partial_t \theta + (u \cdot \nabla)\theta = 0, \\
\mathrm{div} u = 0.
\end{cases} \tag{1.1}
$$

Here the constant ν_1 is the viscous dissipation, $u = (u_1, u_2, u_3)$ is the velocity field, θ is a scalar that may be interpreted physically as thermal (or density,), P is the pressure and $e_3 := (0, 0, 1)^T$.

The system (1.1) is a coupled system of the incompressible Navier-Stokes equations and a scalar convection-diffusion equation. Let $\theta \equiv 0$, then the system (1.1) will degenerate to the incompressible Navier-Stoke equations. The global existence of smooth solutions for the three-dimensional incompressible Navier-Stokes equations with large data is one of the most outstanding open problems, although the two dimensional case has been solved very well. One main challenge for the three dimensional case is the effect of vortex stretching, which is not well understood even in the two-dimensional case([16]). Many literatures have attempted to prove the global regularity for the solution of 3D incompressible Navier-Stokes equations with additional and reasonable conditions. In

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[19], the authors presented the global regularity for the solution of three-dimensional incompressible Navier-Stokes equations on a thin domain. Subsequently, all kinds of boundary conditions for the bounded thin domain cases have been studied by [21, 8, 9, 10]. (See [18] for more details on this issue).

To understand the vortex stretching effect of 3D flows, the Boussinesq system is a classical model because it shares a similar vortex stretching effect.

The system (1.1) is the Navier-Stokes equations coupled with a transport equation, which is named as a partial viscosity system, since there is no viscosity term on θ. So one needs some very careful estimates to figure out such a coupling. See for example [12, 13, 14].

There are some global well-posedness results for the system (1.1) in 2D (see [2, 3, 7]). Subsequently, the results of [2, 3, 7] have been extended to the anisotropic case, see [1, 5, 6, 15]. More generalized extensions can be found in [4, 11, 17] et.al. In this paper, we prove some global existence and uniqueness result for the 3D solution in the thin domain. To our knowledge this is the first result for the 3D solution of (1.1). (Right?).

More precisely, The purpose of this paper is to present a global well-posedness result for the system (1.1) on a 3D thin domain with large initial data. Heuristically, based on the 2D results of [2, 7], for the fluid in a thin domain, it is possible to bound and disclose(?) the mechanism of the coupled vertex stretch term. Moreover, we wish this idea can be used later to other systems such as Visco-Elasticity and MHD systems and the dynamics mechanism for the coupled system on a thin domain.

We study the 3D system (1.1) on the domain:

$$\Omega_\varepsilon \subset R^3, \tag{1.2}$$

where $\Omega_\varepsilon = \Omega \times [0, \varepsilon]$ and $\Omega \subset R^2$ with smooth boundary. Furthermore, we impose the following initial conditions:

$$\begin{cases} (u, \theta)|_{t=0} = (u_0, \theta_0) \\ \text{div} u_0 = 0, \end{cases} \tag{1.3}$$

and boundary conditions:

$$\text{curl} u \times \vec{n} = 0, u \cdot \vec{n} = 0 \ on \ \partial\Omega_\varepsilon. \tag{1.4}$$

We shall prove there exists a small constant ε_0, which depends on the size of the initial data, such that for any $\varepsilon \in (0, \varepsilon_0)$, the system (1.1)-(1.4) with large initial data is global well-posed. Here is our main result.

Theorem 1.1. *Assume the initial data* $(u_0, \theta_0) \in H^2(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$, *for some* $0 < q < 1$, *let* $R_0^2(\varepsilon) := \|u_0\|^2_{H^1(\Omega_\varepsilon)} + \|\theta_0\|^2_{L^2(\Omega_\varepsilon)}$ *satisfy*

$$\lim_{\varepsilon \to 0} \varepsilon^q R_0^2(\varepsilon) = 0 \tag{1.5}$$

then, there exists a constant $\varepsilon_0 > 0$, *such that* $\forall \varepsilon \in [0.\varepsilon_0)$, *the system* (1.1)-(1.4) *admits a unique pair of global solution*

$$(u, \theta) \in L^\infty(0, \infty; H^2(\Omega_\varepsilon)) \times L^\infty(0, \infty; H^1(\Omega_\varepsilon)). \tag{1.6}$$

Remark 1.2. *The condition* (1.5) *implies that for* ε *small enough, the initial data can be picked very large, therefore, our results is a large data result when one considers the thin domain case. On the other hand, if we take initial data small enough, then* ε_0 *can be taken arbitrary large, which means our result applies for the* **wide domain** *instead of thin domain in* $\mathbb{R}^3$.

Remark 1.3. *The case with only horizontal dissipation or vertical dissipations just like the case in* [5, 6, 1] *has also been tried. However, in our case, when* $\varepsilon > 0$*, we cannot bound the term* $\partial_3 u_1$ *and* $\partial_3 u_2$ *even for* ε *small enough. It is an obvious difference between two and three dimensional cases, which also implies that there is obvious difference between the results of* [2, 7] *and our paper although we take* ε *small enough. I suggest to remove this remark, or at least move it later. This does not give you any credit.*

Any new idea worth mentioning here in the proof of the theorem 1.1???

Moreover, we also studied the asymptotic behavior of the solution when the thickness $\varepsilon \to 0$. To interpret our results clearer, we first recall the average operator notations. Let ϕ be a function defined on a 3D thin domain Ω_ε, then the operator M and N are defined as:

$$M\phi = M(\phi_1, \phi_2, \phi_3) = \Big(\frac{1}{\varepsilon}\int_0^\varepsilon \phi_1 dx_3, \frac{1}{\varepsilon}\int_0^\varepsilon \phi_2 dx_3, 0\Big) \tag{1.7}$$

and

$$N\phi = (I - M)\phi. \tag{1.8}$$

Then we have:

Theorem 1.4. *Under the assumptions of Theorem 1.1, for the initial data* $(u_0, \theta_0) \in H^2(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$, $0 < \varepsilon < \varepsilon_0$*, assuming*

$$\lim_{\varepsilon\to 0}(Mu_0, M\theta_0) = (\tilde{v}_0, \tilde{\theta}_0), \ \text{weak in } H^2(\Omega) \times H^1(\Omega), \tag{1.9}$$

then for the solution (u, θ) *of system* (1.1)-(1.4)*, there holds:*

$$\lim_{\varepsilon\to 0}(\|Mu - \tilde{v}\|_{L^2(0,T;H^1(\Omega))} + \|M\theta - \tilde{\theta}\|_{L^\infty(0,T;L^2(\Omega))}) = 0, \tag{1.10}$$

where $(\tilde{v}(x'), \tilde{\theta}(x'))$ *satisfies the following 2D system*

$$\begin{cases} \partial_t \tilde{v} - \nu\Delta'\tilde{v} + (\tilde{v}\cdot\nabla')\tilde{v} + \nabla'\tilde{P} = 0, & (x', t) \in \Omega \times [0, T], \\ \partial_t\tilde{\theta} + (\tilde{u}\cdot\nabla')\tilde{\theta} = 0, \\ \nabla'\cdot\tilde{v} = 0, \\ t = 0, \quad \tilde{v} = \tilde{v}_0(x'), \ \tilde{\theta} = \tilde{\theta}_0(x'). \end{cases} \tag{1.11}$$

Here and hereafter, we use the following notations:

$$\nabla' = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, 0\Big), \ \Delta' = \Big(\frac{\partial^2}{\partial {x_1}^2}, \frac{\partial^2}{\partial {x_2}^2}, 0\Big), x' = (x_1, x_2). \tag{1.12}$$

Our proof basically follows [9, 10] (for 2D case?, or else. If you do not state clearly, the following compare does not make sense.), but there are still some new aspects worth mentioning. First, the structure of the coupled system (1.1) in 3D is much complicated than the 2D case. Even in 2D case, let $w = curl u$ and $U = (\partial_2\theta, -\partial_1\theta)$, we have

$$\begin{cases} \partial_t w + (u\cdot\nabla)w - \nu\Delta w = \partial_1\theta, \\ \partial_t U + (u\cdot\nabla)U = (\nabla u)\cdot U. \end{cases} \tag{1.13}$$

The system (1.13) shares the similar vortex stretching structure with the 3D incompressible Navier-Stokes equations. Besides, we consider the problem (1.1) without viscous on θ, it needs carefully dealt with θ when we get the higher order derivative norm of u.(This paragraph is not very organized. Need to be rewritten.)

This paper is organized as follows: in Section 2, we shall present some preliminary results for the averaging operators and Sobolev-type inequalities on the thin domain, as well as the local well-posedness for the system (1.1)-(1.4). In Section 3, Theorem 1.1 will be proved, and then in Section 4, we shall prove the theorem 1.4. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \le CB$ with an uniform constant C.

2. Preliminary

At the beginning, we shall recall some known properties for the average operator M. The following Lemma can be verified. (See details in [19] and [20]).

Lemma 2.1. *Let the operators M and N be defined as* (1.7) *and* (1.8)*, then there hold:*

(1) *M is an orthogonal projector from $L^2(\Omega_\varepsilon)$ onto $L^2(\Omega)$*
(2) $M^2u = Mu, N^2u = Nu, MN = 0,$
(3) $M\nabla' = \nabla' M, N\nabla' = \nabla' N, M\Delta u = \Delta Mu, N\Delta u = \Delta Nu,$
(4) $\int_{\Omega_\varepsilon} \nabla Nu\cdot\nabla Mudx = 0, \nabla'\cdot Mu = 0,$ *where* $\nabla' = (\partial_1, \partial_2, 0),$
(5) $\|u\|_{L^2(\Omega_\varepsilon)} = \|Mu\|_{L^2(\Omega_\varepsilon)} + \|Nu\|_{L^2(\Omega_\varepsilon)}$, $\|u\|_{H^1(\Omega_\varepsilon)} = \|Mu\|_{H^1(\Omega_\varepsilon)} + \|Nu\|_{H^1(\Omega_\varepsilon)}.$

Lemma 2.2. *For a function $\phi \in H^1(\Omega_\varepsilon)$ satisfying $\phi(x_1,x_2,x_3)|_{x_3=0,\varepsilon} = 0$ or $\int_0^\varepsilon \phi(x_1,x_2,x_3)dx_3 = 0$, we have*

$$\|\phi\|_{L^2(\Omega_\varepsilon)} \le \varepsilon\|\frac{\partial\phi}{\partial x_3}\|_{L^2(\Omega_\varepsilon)}. \tag{2.1}$$

For any vector $u \in H^2(\Omega_\varepsilon)$ satisfying boundary condition (1.4)*, we have*

$$\|u\|_{L^\infty(\Omega_\varepsilon)} \le C\|u\|^{\frac14}_{L^2(\Omega_\varepsilon)}(\Sigma^3_{i,j=1}(\|\frac{\partial^2 u}{\partial x_i\partial x_j}\|^{\frac34}_{L^2\Omega_\varepsilon)}), \tag{2.2}$$

and for $2 \le q \le 6$, there holds

$$\|u\|^2_{L^q(\Omega_\varepsilon)} \le C\varepsilon^{\frac{6-q}{2q}}\|\frac{\partial u}{\partial x_3}\|^2_{L^2(\Omega_\varepsilon)}. \tag{2.3}$$

To prove our results, we give the following local well posedness results for system (1.1)-(1.4) beforehand.

Proposition 2.3. *(Local Well Posedness) Let $(u_0,\theta_0) \in H^2(\Omega_\varepsilon)\times H^1(\Omega_\varepsilon)$, then there exists a $T^*(\varepsilon, \|u_0\|_{H^2(\Omega_\varepsilon)}, \|\theta_0\|_{H^1(\Omega_\varepsilon)}) > 0$, such that the system* (1.1)-(1.4) *admits a unique pair of strong solution*

$$(u,\theta) \in L^\infty(0,T^*;H^2(\Omega_\varepsilon)\times L^\infty(0,T^*;H^1(\Omega_\varepsilon)), \tag{2.4}$$

and

$$\|u\|^2_{L^\infty(0,T^*;H^2(\Omega_\varepsilon))} + \|\theta\|^2_{L^\infty(0,T^*;H^1(\Omega_\varepsilon))} \le C(\|u_0\|^2_{H^2} + \|\theta_0\|^2_{H^1}). \tag{2.5}$$

Proof. The local well posedness for the system (1.1)-(1.4) can be verified by standard procedure, see for example [20]. For completely, we give a brief proof as follows.

Step 1. Estimates for $\|u_t\|_{L^2(\Omega_\varepsilon)}$ and $\|\nabla u\|_{L^2(\Omega_\varepsilon)}$.
By energy estimates, we first get

$$\|\theta\|_{L^p(\Omega_\varepsilon)} \le \|\theta_0\|_{L^p(\Omega_\varepsilon)}, \ \ for\ any\ 1 \le p \le \infty, \tag{2.6}$$

and

$$\frac{1}{2}\|u\|^2_{L^2(\Omega_\varepsilon)} + \nu \int_0^t \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \lesssim \|\theta_0\|^2_{L^2(\Omega_\varepsilon)} + \|u_0\|^2_{L^2(\Omega_\varepsilon)}. \tag{2.7}$$

Noting that $\nabla \cdot u = 0$ and $\Delta u = -\nabla \times (\nabla \times u)$, by testing with $\nabla \times (\nabla \times u)$ to the first equation of (1.1), we have

$$\int_{\Omega_\varepsilon} (\partial_t u - \nu \Delta u + (u\cdot\nabla)u + \nabla P)\cdot[\nabla \times (\nabla \times u)]dx = \int_{\Omega_\varepsilon} \theta e_3 \cdot [\nabla \times (\nabla \times u)]dx. \tag{2.8}$$

Integrating by parts, and using the multiplicative inequality

$$\|\nabla u\|_{L^3(\Omega_\varepsilon)} \le \|\nabla u\|^{\frac{1}{2}}_{L^2(\Omega_\varepsilon)}\|\nabla^2 u\|^{\frac{1}{2}}_{L^2(\Omega_\varepsilon)}, \tag{2.9}$$

from (2.8), we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla \times u\|^2_{L^2(\Omega_\varepsilon)} + \frac{\nu}{4}\|\Delta u\|^2_{L^2(\Omega_\varepsilon)} \lesssim \|\theta_0\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla u\|^3_{L^2(\Omega_\varepsilon)}. \tag{2.10}$$

By testing with u_t to the first equation of $(1.1)_1$and integrating on Ω_ε we have

$$\int_{\Omega_\varepsilon} (u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla P)u_t dx = \int_{\Omega_\varepsilon} \theta e_3 u_t dx. \tag{2.11}$$

Integrating by parts, we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |u_t|^2 dx + \frac{\nu}{2}\frac{d}{dt}\int_{\Omega_\varepsilon} |\nabla\times u|^2 dx &= -\int_{\Omega_\varepsilon} (u\cdot\nabla)u\cdot u_t dx + \int_{\Omega_\varepsilon} \theta e_3 \cdot u_t dx, \\ &\lesssim \int_{\Omega_\varepsilon} |\theta|^2 dx + \int_{\Omega_\varepsilon} |(u\cdot\nabla)u|^2 dx + \frac{1}{4}\int_{\Omega_\varepsilon} |u_t|^2 dx \\ &\lesssim \int_{\Omega_\varepsilon} |\theta|^2 dx + (\int_{\Omega_\varepsilon} |u|^6 dx)^{\frac{1}{3}}(\int_{\Omega_\varepsilon} |\nabla u|^3 dx)^{\frac{2}{3}}) \lesssim \|\theta_0\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla u\|^6_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{2.12}$$

Noting that $\nabla \cdot u = 0$ and from (2.12), one gets

$$\int_0^t (\|u_t\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla^2 u\|^2_{L^2(\Omega_\varepsilon)})ds + \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \le C(t)(1 + \int_0^t \|\nabla u\|^6_{L^2(\Omega_\varepsilon)} ds), \tag{2.13}$$

and

$$\sup_{0\le s\le T} \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \le C(T, \|\nabla u_0\|_{L^2(\Omega_\varepsilon)}, \|\theta_0\|_{L^2(\Omega_\varepsilon)}) \exp(C\int_0^T \|\nabla u\|^4_{L^2(\Omega_\varepsilon)} ds). \tag{2.14}$$

Step 2. Higher order estimates: $\|\nabla u_t\|_{L^2(\Omega_\varepsilon)}$, $\|\nabla^2 u\|_{L^6(\Omega_\varepsilon)}$ and $\|\theta\|_{H^1(\Omega_\varepsilon)}$. Applying ∂_t to $(1.1)_1$ and then testing with u_t, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_\varepsilon}|u_t|^2dx+\nu\int_{\Omega_\varepsilon}|\nabla\times u_t|^2dx &\le \int_{\Omega_\varepsilon}\theta_t e_3\cdot u_t dx+\int_{\Omega_\varepsilon}|u_t||\nabla u_t||u|dx,\\ &\le \int_{\Omega_\varepsilon}|u||\nabla u_t||\theta|dx+\int_{\Omega_\varepsilon}|u_t||\nabla u_t||u|dx,\\ &\le C(t)(\|\theta\|^2_{L^3(\Omega_\varepsilon)}\|\nabla u\|^2_{L^2(\Omega_\varepsilon)}+\|\nabla u\|^4_{L^2(\Omega_\varepsilon)}\|u_t\|^2_{L^2(\Omega_\varepsilon)}),\end{aligned}\tag{2.15}$$

which implies

$$\frac{d}{dt}\|u_t\|^2_{L^2(\Omega_\varepsilon)}+\nu\|\nabla u_t\|^2_{L^2(\Omega_\varepsilon)}\lesssim\|\theta_0\|^2_{H^1(\Omega_\varepsilon)}\|\nabla u\|^2_{L^2(\Omega_\varepsilon)}+\|\nabla u\|^4_{L^2(\Omega_\varepsilon)}\|u_t\|^2_{L^2(\Omega_\varepsilon)}.\tag{2.16}$$

By Gronwall's inequality and then from (2.15)-(2.16), we have

$$\begin{aligned}\|u_t\|^2_{L^2(\Omega_\varepsilon)}+\|\nabla u\|^2_{L^2(\Omega_\varepsilon)}+\int_0^t(\|\nabla u_t\|^2_{L^2(\Omega_\varepsilon)}+\|u_t\|^2_{L^2(\Omega_\varepsilon)}+\|\nabla^2 u\|^2_{L^2(\Omega_\varepsilon)})ds\\ \le C(t)(1+\exp(C\int_0^t\|\nabla u\|^4_{L^2(\Omega_\varepsilon)}ds)),\end{aligned}\tag{2.17}$$

Now, by the equation (1.1), we have

$$\begin{aligned}\|\nabla^2 u\|^2_{L^2(\Omega_\varepsilon)}+\|\nabla P\|^2_{L^2(\Omega_\varepsilon)}&\lesssim\|u_t\|^2_{L^2(\Omega_\varepsilon)}+\|u\cdot\nabla u\|^2_{L^2(\Omega_\varepsilon)}+\|\theta\|^2_{L^2(\Omega_\varepsilon)}\\ &\le C(t)((1+\|u_t\|^2_{L^2(\Omega_\varepsilon)}+\|\nabla u\|^6_{L^2(\Omega_\varepsilon)}).\end{aligned}\tag{2.18}$$

By using Lemma 2.1, we have

$$\begin{aligned}\|\nabla^2 u\|^2_{L^6(\Omega_\varepsilon)}+\|\nabla P\|^2_{L^6(\Omega_\varepsilon)}&\le\|u_t\|^2_{L^6(\Omega_\varepsilon)}+\|u\|^2_{L^\infty(\Omega_\varepsilon)}\|\nabla u\|^2_{L^6(\Omega_\varepsilon)}+\|\theta\|^2_{L^6(\Omega_\varepsilon)}\\ &\le C(t)(1+\|\nabla u_t\|^2_{L^2(\Omega_\varepsilon)})+\|u\|^{\frac{1}{2}}_{L^2(\Omega_\varepsilon)}\|\Delta u\|^{\frac{7}{2}}_{L^2(\Omega_\varepsilon)}.\end{aligned}\tag{2.19}$$

From (2.18) and (2.19), we get

$$\sup_{0\le s\le T}(\|\nabla^2 u\|^2_{L^2(\Omega_\varepsilon)}+\|\nabla P\|^2_{L^2(\Omega_\varepsilon)})\le C(t)(1+\exp(C\int_0^T\|\nabla u\|^4_{L^2(\Omega_\varepsilon)}ds)),\tag{2.20}$$

and

$$\int_0^T\|\nabla^2 u\|^2_{L^6(\Omega_\varepsilon)}+\|\nabla P\|^2_{L^6(\Omega_\varepsilon)}ds\le C(t)(1+\exp(C\int_0^T\|\nabla u\|^4_{L^2(\Omega_\varepsilon)}ds)).\tag{2.21}$$

Applying ∇ to equation $(1.1)_2$, and testing the equation by $\nabla\theta$, one gives

$$\frac{d}{dt}\int_\Omega|\nabla\theta|^2dx\le\int_\Omega|\nabla u||\nabla\theta|^2dx.\tag{2.22}$$

Hence, from (2.22) and (2.6), we have

$$\begin{aligned}\sup_{0\le s\le T}\|\theta\|^2_{H^1(\Omega_\varepsilon)}&\le\|\theta_0\|^2_{H^1(\Omega_\varepsilon)}\exp(C\int_0^T\|\nabla u\|_{W^{1,6}(\Omega_\varepsilon)}ds)\\ &\le\|\theta_0\|^2_{H^1(\Omega_\varepsilon)}\exp(C\int_0^T\|\nabla u\|^4_{L^2(\Omega_\varepsilon)}ds).\end{aligned}\tag{2.23}$$

Using this uniform bound (2.20) and (2.23), by the standard fixed point theory, it is easy to verify that there exists a small time $T^*(u_0, \theta_0)$, such that the system (1.1)-(1.4) admits a unique pair of local solution $(u, \theta) \in L^\infty(0, T^*; H^2(\Omega_\varepsilon)) \times L^\infty(0, T^*; H^1(\Omega_\varepsilon))$.

□

Remark 2.4. *And we also get the following fact: for a suitable constant $\sigma > 0$, and let R_0 be defined as in theorem 1.1, then $\exists T^\sigma > 0$ such that $\|\nabla u(t, \cdot)\|^2_{L^2(\Omega_\varepsilon)} + \|\theta(t, \cdot)\|^2_{L^2(\Omega_\varepsilon)} \leq \sigma R_0^2,\ \forall\ 0 \leq t \leq T^\sigma$. Moreover, if $T^\sigma < \infty$ is the maximal time satisfying the above inequality, then*

$$\|\nabla u(T^\sigma, \cdot)\|^2_{L^2(\Omega_\varepsilon)} + \|\theta(T^\sigma, \cdot)\|^2_{L^2(\Omega_\varepsilon)} = \sigma R_0^2. \tag{2.24}$$

3. Proof of Theorem 1.1.

We shall prove the global well-posedness by using the bootstrap procedure. That is to say we need prove the well-posedness on a short time interval $[0, T_0]$ with T_0 small enough. By verifying the data $(v(\cdot, t_\varepsilon), \theta(\cdot, t_\varepsilon))$ satisfying the same size as $t = 0$, then we set the data $t = t_\varepsilon$ as the initial data and we can extend the local solution on $[t_\varepsilon, 2t_\varepsilon]$. Repeating the procedure, we can get the global wellposedness. To achieve our purpose, we need to use the thickness of the domain much carefully.

Since θ satisfies a transport equation, it is obviously that

$$\|M\theta\|_{L^2(\Omega_\varepsilon)}, \|N\theta\|_{L^2(\Omega_\varepsilon)} \leq \|\theta\|_{L^2(\Omega_\varepsilon)}. \tag{3.1}$$

Proposition 3.1. *Assume the system (1.1)-(1.4) satisfies the same assumptions as in theorem 1.1, then there hold the following estimates:*

$$\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)} \leq \|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)} \exp(-\frac{\nu t}{2\varepsilon^2}) + \frac{t}{\nu}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)} \exp(-\frac{\nu t}{2\varepsilon^2}), \tag{3.2}$$

$$\|\nabla Mu\|^2_{L^2(\Omega_\varepsilon)} \lesssim \|u_0\|^2_{L^2(\Omega_\varepsilon)} \exp(-\frac{\nu\lambda_1 t}{2}) + \varepsilon R_0^4 \cdot (1 + t) \exp(-\frac{\nu\lambda_1 t}{2}), \tag{3.3}$$

and

$$\int_0^t \|\nabla^2 Mu\|^2_{L^2(\Omega_\varepsilon)} ds + \int_0^t \|\nabla^2 Nu\|^2_{L^2(\Omega_\varepsilon)} ds \leq \frac{2}{\nu^2}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)} t + \frac{2}{\nu}\|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)}, \tag{3.4}$$

as well as

$$\int_0^t \|\nabla Mu\|^2_{L^2(\Omega_\varepsilon)} ds + \int_0^t \|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)} ds \leq \frac{2\varepsilon^2}{\nu^2}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)} t + \frac{2\varepsilon^2}{\nu}\|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.5}$$

Where R_0 is defined in theorem 1.1, and $\lambda_1 > 0$ is the first eigenvalue of operator $-\Delta'$.

Proof. We shall prove this proposition in the following steps.

Step 1. H^1 Estimates for Nu .
To estimate Nu, multiplying the equations $(1.1)_1$ by $-\Delta Nu$, thanks to Lemma 2.1 and

(1.4), we have:

$$\begin{aligned}
(3.6)\qquad &\frac{d}{dt}\int_{\Omega_\varepsilon} Nu\cdot N(-\Delta u)dx+\nu\int_{\Omega_\varepsilon}|N\Delta u|^2dx\\
&+\int_{\Omega_\varepsilon}(Mu\cdot\nabla)Nu\cdot N(-\Delta u)dx+\int_{\Omega_\varepsilon}(Nu\cdot\nabla)Mu\cdot N(-\Delta u)dx\\
&+\int_{\Omega_\varepsilon}(Nu\cdot\nabla)Nu\cdot N(-\Delta u)dx+\int_{\Omega_\varepsilon}(Mu\cdot\nabla)Mu\cdot N(-\Delta u)dx\\
&=\int_{\Omega_\varepsilon}\theta e_3\cdot N(-\Delta u)dx.
\end{aligned}$$

From Lemma 2.1, we have the following fact

$$(3.7)\qquad \int_{\Omega_\varepsilon}(Mu\cdot\nabla)Mu\cdot N(\Delta u)dx=\int_\Omega(Mu\cdot\nabla')Mu\cdot(\int_0^\varepsilon N(\Delta u)dx_3)dx'=0,$$

then, from (3.6) we have

$$\begin{aligned}
(3.8)\qquad &\frac{1}{2}\frac{d}{dt}\int_{\Omega_\varepsilon}|\nabla Nu|^2dx+\frac{\nu}{2}\int_{\Omega_\varepsilon}|\Delta Nu|^2dx\\
&\lesssim\int_{\Omega_\varepsilon}|N\theta|^2dx+\int_{\Omega_\varepsilon}|(Mu\cdot\nabla)Nu\cdot N(\Delta u)|dx\\
&\quad+\int_{\Omega_\varepsilon}|(Nu\cdot\nabla)Mu\cdot N(\Delta u)|dx+\int_{\Omega_\varepsilon}|(Nu\cdot\nabla)Nu\cdot N(\Delta u)|dx\\
&:=\frac{1}{2\nu}\int_{\Omega_\varepsilon}|N\theta|^2dx+I_1+I_2+I_3.
\end{aligned}$$

By lemma 2.2, for any $2<p\le 4$, let $\frac{q}{2}=\frac{2}{p}-\frac{1}{2}$, we have

$$\begin{aligned}
(3.9)\quad I_1\le\int_{\Omega_\varepsilon}|Mu||\nabla Nu||\Delta Nu|dx&\lesssim\|Mu\|_{L^{\frac{2p}{p-2}}(\Omega_\varepsilon)}\|\nabla Nu\|_{L^p(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}\\
&\lesssim\varepsilon^{\frac{p-2}{2p}}\|\nabla' Mu\|_{L^2(\Omega)}\cdot\varepsilon^{\frac{6-p}{2p}}\|\nabla^2Nu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}\\
&\lesssim\varepsilon^{\frac{2}{p}-\frac{1}{2}}\|\nabla Mu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)},
\end{aligned}$$

and

$$\begin{aligned}
(3.10)\quad I_2\le\int_{\Omega_\varepsilon}|Nu||\nabla Mu||\Delta Nu|dx&\lesssim\|Nu\|_{L^\infty(\Omega_\varepsilon)}\|\nabla Mu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}\\
&\lesssim\varepsilon^{\frac14}\|\nabla Nu\|^{\frac14}_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|^{\frac34}_{L^2(\Omega_\varepsilon)}\|\nabla Mu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}\\
&\lesssim\varepsilon^{\frac12}\|\nabla Mu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|^2_{L^2(\Omega_\varepsilon)},
\end{aligned}$$

as well as

$$\begin{aligned}
(3.11)\quad I_3\le\int_{\Omega_\varepsilon}|Nu||\nabla Nu||\Delta Nu|dx&\lesssim\|Nu\|_{L^6(\Omega_\varepsilon)}\|\nabla Nu\|_{L^3(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}\\
\lesssim\varepsilon^{\frac12}\|\nabla Nu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}&\lesssim\varepsilon^{\frac12}\|\nabla Nu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|^2_{L^2(\Omega_\varepsilon)}.
\end{aligned}$$

Combining the inequalities (3.8)-(3.11) and from (3.6), we get

$$\frac{d}{dt}\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)} + [\nu - C\varepsilon^{\frac{q}{2}}(\|\nabla Mu\|_{L^2(\Omega_\varepsilon)} + \|\nabla Nu\|_{L^2(\Omega_\varepsilon)})]\|\Delta Nu\|^2_{L^2(\Omega_\varepsilon)} \leq \frac{1}{\nu}\|N\theta\|^2_{L^2(\Omega_\varepsilon)}, \tag{3.12}$$

where $0 < q < 1$. Then by (2.24) and noting the assumption: $\lim_{\varepsilon\to 0}\varepsilon^q R_0^2(\varepsilon) = 0$, we can choose ε small enough, such that

$$\frac{\nu}{2} \leq [\nu - C\varepsilon^{\frac{q}{2}}(\|\nabla Mu\|_{L^2(\Omega_\varepsilon)} + \|\nabla Nu\|_{L^2(\Omega_\varepsilon)})]. \tag{3.13}$$

Noting that

$$\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)} \leq \varepsilon^2\|\nabla^2 Nu\|^2_{L^2(\Omega_\varepsilon)}, \|N\theta\|^2_{L^2(\Omega_\varepsilon)} \leq \|\theta_0\|^2_{L^2(\Omega_\varepsilon)}, \tag{3.14}$$

we can rewrite (3.12) as

$$\frac{d}{dt}\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)} + \frac{\nu}{2\varepsilon^2}\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)} \leq \frac{1}{\nu}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.15}$$

At last, by Gronwall's inequality, we have

$$\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)} \leq \|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)}\exp(-\frac{\nu t}{2\varepsilon^2}) + \frac{t}{\nu}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)}\exp(-\frac{\nu t}{2\varepsilon^2}), \tag{3.16}$$

and

$$\int_0^t \|\nabla^2 Nu\|^2_{L^2(\Omega_\varepsilon)}ds \lesssim \frac{2}{\nu^2}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)}t + \frac{2}{\nu}\|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)}, \tag{3.17}$$

as well as

$$\int_0^t \|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)}ds \leq \frac{2\varepsilon^2}{\nu^2}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)}t + \frac{2\varepsilon^2}{\nu}\|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.18}$$

Step 2. H^1 Estimates for Mu .
By Lemma 2.1, we have

$$\int_{\Omega_\varepsilon}(Mu\cdot\nabla)Mu\cdot Mudx = \frac{1}{2}\int_{\Omega_\varepsilon}(\nabla\cdot Mu)|Mu|^2dx = 0, \tag{3.19}$$

$$\int_{\Omega_\varepsilon}(Nu\cdot\nabla)Mu\cdot Mudx = \frac{1}{2}\int_{\Omega_\varepsilon}(\nabla\cdot Nu)|Mu|^2dx = 0, \tag{3.20}$$

$$\int_{\Omega_\varepsilon}(Mu\cdot\nabla)Mu\cdot Nudx = \int_{\Omega_\varepsilon}(Mu\cdot\nabla)Mu\cdot\int_0^\varepsilon Nudx_3dx' = 0. \tag{3.21}$$

Multiplying the equations $(1.1)_1$ by Mu and integrating on Ω_ε, noting that $Mu = (Mu_1, Mu_2, 0)$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_\varepsilon}|Mu|^2dx + \nu\int_{\Omega_\varepsilon}|\nabla Mu|^2dx &\lesssim |\int_{\Omega_\varepsilon}\theta e_3\cdot Mudx| + \int_{\Omega_\varepsilon}|(Nu\cdot\nabla)Nu\cdot Mu|dx \\ &\lesssim \|Nu\|_{L^6(\Omega_\varepsilon)}\|\nabla Nu\|_{L^3(\Omega_\varepsilon)}\|Mu\|_{L^2(\Omega_\varepsilon)} \\ &\lesssim \varepsilon^{\frac{1}{2}}\|\nabla Nu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}\|Mu\|_{L^2(\Omega_\varepsilon)}.\end{aligned} \tag{3.22}$$

In the last inequality, we used Lemma 2.2.

Noting that the domain $\Omega \subset R^2$ is a bounded domain with smooth boundary and the boundary condition (1.4), then we have the following generalized Poincare inequatlity.

$$\|Mu\|^2_{L^2(\Omega_\varepsilon)} \le \frac{1}{\lambda_1}\|\nabla' Mu\|^2_{L^2(\Omega_\varepsilon)}, \tag{3.23}$$

where λ_1 is first eigenvalue of the operator $-\Delta'$.

Then by Gronwall's inequality, (3.16)-(3.18) and (3.22)-(3.23), we have

$$\begin{aligned}\|Mu\|^2_{L^2(\Omega_\varepsilon)} \lesssim{}& \exp(-\frac{\nu\lambda_1 t}{2})[\|Mu_0\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon(\|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)} + \frac{2\varepsilon^2}{\nu^2}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)}) \\ &\cdot(\frac{2}{\nu^2}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)}t + \frac{2}{\nu}\|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)})] \\ &\lesssim \|u_0\|^2_{L^2(\Omega_\varepsilon)}\exp(-\frac{\nu\lambda_1 t}{2}) + \varepsilon R_0^4\cdot(1+t)\exp(-\frac{\nu\lambda_1 t}{2}),\end{aligned} \tag{3.24}$$

where R_0 is defined in theorem 1.1.

Replacing (3.24) to (3.22), we also have

$$\int_0^t \|\nabla Mu\|^2_{L^2(\Omega_\varepsilon)}ds \le \frac{2\lambda_1}{\nu^2}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)}t + \frac{2\lambda_1}{\nu}\|Mu_0\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon R_0^4\cdot(1+t). \tag{3.25}$$

Multiplying the equations $(1.1)_1$ by $-M\Delta' u$, noting that $Mu = (Mu_1, Mu_2, 0)$, we have:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega_\varepsilon}|\nabla' Mu|^2dx + \nu\int_{\Omega_\varepsilon}|\Delta' Mu|^2dx \\ &-\int_{\Omega_\varepsilon}(Mu\cdot\nabla)Mu\cdot M\Delta' udx - \int_{\Omega_\varepsilon}(Nu\cdot\nabla)Mu\cdot M\Delta' udx \\ &-\int_{\Omega_\varepsilon}(Nu\cdot\nabla)Nu\cdot M\Delta' udx - \int_{\Omega_\varepsilon}(Mu\cdot\nabla)Nu\cdot M\Delta' udx = 0.\end{aligned} \tag{3.26}$$

Due to Lemma 2.1, there hold

$$\int_{\Omega_\varepsilon}(Nu\cdot\nabla)Mu\cdot M(A_\varepsilon u)dx = \int_{R^2}(\int_0^\varepsilon Nudx_3\cdot\nabla')Mu\cdot M(A_\varepsilon u)dx' = 0, \tag{3.27}$$

$$\int_{\Omega_\varepsilon}(Mu\cdot\nabla)Nu\cdot M(A_\varepsilon u)dx = \int_{R^2}\int_0^\varepsilon(Mu\cdot N\nabla)u\cdot M(A_\varepsilon u)dx_3dx' = 0. \tag{3.28}$$

As to the third term on the left hand side of (3.26), we have,

$$\begin{aligned}&\int_{\Omega_\varepsilon}(Mu\cdot\nabla)Mu\cdot M(\Delta u)dx = \int_{\Omega_\varepsilon}(Mu\cdot\nabla)Mu\cdot M(\Delta' u)dx \\ &= \frac{\varepsilon}{2}\int_\Omega \nabla(|Mu|^2)curlcurlMudx' + \varepsilon\int_\Omega(Mu\times curlMu)curlcurlMudx' \\ &= \varepsilon\int_\Omega((\nabla(curl'Mu)\cdot\vec{e}_3)Mu - (\nabla(curl'Mu)\cdot Mu)\vec{e}_3)\cdot(curl'Mu\vec{e}_3)dx' \\ &= \varepsilon\int_\Omega -\frac{1}{2}\nabla|curl'Mu|^2\cdot Mudx' = \varepsilon\int_\Omega\frac{1}{2}|curl'Mu|^2\nabla\cdot Mudx' = 0.\end{aligned} \tag{3.29}$$

Where $curl' = \nabla'\times$. Combining (3.26)-(3.29), we get

$$\frac{d}{dt}\int_{\Omega_\varepsilon}|\nabla' Mu|^2dx+\nu\int_{\Omega_\varepsilon}|\Delta' Mu|^2dx\lesssim\int_{\Omega_\varepsilon}|(Nu\cdot\nabla)Nu\cdot M\Delta' u|dx. \tag{3.30}$$

Similar to (3.11), by using Lemma 2.2, we get

$$\begin{aligned}\int_{\Omega_\varepsilon}|(Nu\cdot\nabla)Nu\cdot M(\Delta' u)|dx&\le\|Nu\|_{L^6(\Omega_\varepsilon)}\|\nabla Nu\|_{L^3(\Omega_\varepsilon)}\|\Delta' Mu\|_{L^2(\Omega_\varepsilon)}\\&\le C\varepsilon^{\frac12}\|\nabla Nu\|_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|_{L^2(\Omega_\varepsilon)}\|\Delta' Mu\|_{L^2(\Omega_\varepsilon)}\\&\le C\varepsilon\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|^2_{L^2(\Omega_\varepsilon)}+\frac{\nu}{4}\|\Delta' Mu\|^2_{L^2(\Omega_\varepsilon)},\end{aligned} \tag{3.31}$$

then, combining the inequalities (3.26), (3.30) and (3.31), we have

$$\frac{d}{dt}\|\nabla Mu\|^2_{L^2(\Omega_\varepsilon)}+\frac{\nu}{4}\|\Delta Mu\|^2_{L^2(\Omega_\varepsilon)}\lesssim\varepsilon\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.32}$$

Again, using the generalized Poincare inequality, there holds

$$\|\nabla' Mu\|^2_{L^2(\Omega_\varepsilon)}\le\frac{1}{\lambda_1}\|\Delta' Mu\|^2_{L^2(\Omega_\varepsilon)}, \tag{3.33}$$

then we can write (3.32) as

$$\frac{d}{dt}\|\nabla Mu\|^2_{L^2(\Omega_\varepsilon)}+\frac{\nu\lambda_1}{2}\|\nabla Mu\|^2_{L^2(\Omega_\varepsilon)}\lesssim\varepsilon\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)}\|\Delta Nu\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.34}$$

Recalling (3.16)-(3.17), we have

$$\begin{aligned}\|\nabla Mu\|^2_{L^2(\Omega_\varepsilon)}&\lesssim\exp(-\frac{\nu\lambda_1 t}{2})[\|\nabla Mu_0\|^2_{L^2(\Omega_\varepsilon)}+\varepsilon(\|\nabla Nu_0\|^2_{L^2(\Omega_\varepsilon)}+\|N\theta_0\|^2_{L^2(\Omega_\varepsilon)})\\&\qquad\cdot(\frac{2}{\nu^2}\|N\theta_0\|^2_{L^2(\Omega_\varepsilon)}t+\frac{2}{\nu}\|\nabla Nu_0\|^2_{L^2(\Omega_\varepsilon)})]\\&\lesssim\|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)}\exp(-\frac{\nu\lambda_1 t}{2})+\varepsilon R_0^4\cdot(1+t)\exp(-\frac{\nu\lambda_1 t}{2}),\end{aligned} \tag{3.35}$$

and

$$\int_0^t\|\nabla^2 Mu\|^2_{L^2(\Omega_\varepsilon)}ds\lesssim\varepsilon R_0^2(\frac{1}{\nu^2}\|\theta_0\|^2_{L^2(\Omega_\varepsilon)}t+\frac{1}{\nu}\|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)})+\frac{1}{\nu}\|\nabla^2 u_0\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.36}$$

□

In the following, we shall present that the local solution can be extended to globally. We give it by the following Proposition.

Proposition 3.2. *Let* $\lim_{\varepsilon\to0}\varepsilon^q R_0^2=0$, *for some* $0\le q<1$. *Then there exists a* $\varepsilon_1>0$, *for* $0<\varepsilon\le\varepsilon_1$ *such that:*

$$\lim_{\varepsilon\to0}\varepsilon^{1-q}T^*=+\infty. \tag{3.37}$$

Proof.

From the inequalities (3.16) and (3.35), for $0 \le t \le T^*$ we have

$$\begin{aligned} \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} &\le \|\nabla M u\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla N u\|^2_{L^2(\Omega_\varepsilon)} \\ &\le C_0 \|\nabla M u_0\|^2_{L^2(\Omega_\varepsilon)} \exp(-\frac{\nu\lambda_1 t}{2}) + C_0 \|\nabla N u_0\|^2_{L^2(\Omega_\varepsilon)} \exp(-\frac{\nu t}{2\varepsilon^2}) \\ &\quad + C_0 \frac{2\varepsilon^2}{\nu^2} \|N\theta_0\|^2_{L^2(\Omega_\varepsilon)} + C_0 \varepsilon R_0^4 \cdot (1+t) \exp(-\frac{\nu\lambda_1 t}{2}.) \\ &\le \frac{\sigma}{4} R_0^2 + C_0 \varepsilon^q R_0^2 \varepsilon^{1-q} R_0^2 \cdot (1+t) \exp(-\frac{\nu\lambda_1 t}{2}.). \end{aligned} \tag{3.38}$$

where C_0 is an uniform constant, and $\sigma = 4C_0 \max\{1, \frac{2\varepsilon^2}{\nu^2}\} > 0$. Then for ε small enough, from (3.38), we conclude that

$$\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} < \sigma R_0^2, \qquad 0 \le t \le T^*. \tag{3.39}$$

Next, we shall show that T^* can be extended to infinity. Otherwise, suppose that $T^* < \infty$, without loss of generality, we assume

$$\|\nabla u(T^*, \cdot)\|^2_{L^2(\Omega_\varepsilon)} = \sigma R_0^2. \tag{3.40}$$

Then from (3.38), there holds

$$\frac{3\sigma}{4} \le C_0 \varepsilon^q R_0^2 \cdot \varepsilon^{1-q} (1 + T^*). \tag{3.41}$$

So, if (3.37) doesn't hold, it will contradict with $\lim_{\varepsilon\to 0} \varepsilon^q R_0^2 = 0$. □.

Proof. (**The proof of theorem 1.1.**) Recalling Proposition 3.1 and Proposition 3.2, and the assumption $\lim_{\varepsilon\to 0} \varepsilon^q R_0^2 = 0$, there exists a constant ε_1 small enough, such that for any $0 < \varepsilon \le \varepsilon_1$ satisfying :

$$\lim_{\varepsilon\to 0} \varepsilon^{1-q} T^* > 4, \quad \exp(-\frac{\nu\lambda_1}{2\varepsilon^{\frac{1-q}{2}}}) + \frac{2\varepsilon^2}{\nu^2} + \varepsilon^q R_0^2 \le \frac{1}{4C_0}. \tag{3.42}$$

Where C_0 is an uniform constant.

We write $t_\varepsilon = \varepsilon^{\frac{q-1}{2}}$, for any $0 < \varepsilon \le \varepsilon_1$. From (3.42), then the system (1.1)-(1.4) is well-posed in $0 \le t \le t_\varepsilon$.

By Proposition 3.1, on the interval $t_\varepsilon \le t < 2t_\varepsilon$, one has

$$\|\nabla N u\|^2_{L^2(\Omega_\varepsilon)} \le C_0 \|\nabla N u_0\|^2_{L^2(\Omega_\varepsilon)} \exp(-\frac{\nu t}{2\varepsilon^2}) + C_0 \frac{2\varepsilon^2}{\nu^2} \|N\theta_0\|^2_{L^2(\Omega_\varepsilon)} \le \frac{1}{4} R_0^2, \tag{3.43}$$

and

$$\|\nabla M u\|^2_{L^2(\Omega_\varepsilon)} \le C_0 \|\nabla u_0\|^2_{L^2(\Omega_\varepsilon)} \exp(-\frac{\nu\lambda_1 t}{2}) + C_0 \varepsilon R_0^4 \cdot (1+t) \exp(-\frac{\nu\lambda_1 t}{2}) \le \frac{1}{4} R_0^2. \tag{3.44}$$

(3.43) and (3.47) imply that

$$\|\nabla u(2\varepsilon^{\frac{q-1}{2}}, \cdot)\|^2_{L^2(\Omega_\varepsilon)} \le \frac{1}{2} R_0^2. \tag{3.45}$$

Next, we consider the equations (1.1)-(1.4) with the initial data given at $t_0 = 2\varepsilon^{\frac{1-q}{2}}$, and write

$$R_1^2(\varepsilon) := \|\nabla u(2\varepsilon^{\frac{1-q}{2}}, \cdot)\|^2_{L^2(\Omega_\varepsilon)} + \|\theta(2\varepsilon^{\frac{1-q}{2}}, \cdot)\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.46}$$

Then, using Proposition 2.1 again, the system??? (??) with initial data given by (3.46) admits an unique pair of solution on the interval $2t_\varepsilon \le t < 3t_\varepsilon$ and satisfies

$$\|\nabla Nu(t,\cdot)\|^2_{L^2(\Omega_\varepsilon)} \le C_0\|\nabla u(2\varepsilon^{\frac{q-1}{2}},\cdot)\|^2_{L^2(\Omega_\varepsilon)}\exp(-\frac{\nu(t-t_0)}{2\varepsilon^2}) + C_0\frac{2\varepsilon^2}{\nu^2}\|\theta(2\varepsilon^{\frac{1-q}{2}},\cdot)\|^2_{L^2(\Omega_\varepsilon)} \le \frac{1}{4}R_1^2 \le \frac{1}{8}R_0^2, \tag{3.47}$$

and

$$\|\nabla Mu(t,\cdot)\|^2_{L^2(\Omega_\varepsilon)} \le C_0\|\nabla u(2\varepsilon^{\frac{q-1}{2}},\cdot)\|^2_{L^2(\Omega_\varepsilon)}\exp(-\frac{\nu\lambda_1(t-t_0)}{2}) + C_0\varepsilon R_1^4\cdot(1+(t-t_0))\exp(-\frac{\nu\lambda_1(t-t_0)}{2}) \le \frac{1}{4}R_1^2 \le \frac{1}{8}R_0^2. \tag{3.48}$$

Therefore, when $2t_\varepsilon \le t < 3t_\varepsilon$

$$\|\nabla u(t)\|^2_{L^2(\Omega_\varepsilon)} \le \frac{1}{4}R_0^2. \tag{3.49}$$

Repeating this procedure and then we verified that $T^* = \infty,\ for\ any\ 0 < \varepsilon \le \varepsilon_1$.

□

4. The proof of theorem 1.4.

In this section, we shall study the limit equations of (1.1)-(1.4) when $\varepsilon \to 0$.

Proposition 4.1. *Let (u,θ) be a pair of solution to system (1.1)-(1.4) with initial data satisfies the same assumptions as in theorem 1.4, then we have*

$$\begin{cases} \lim_{\varepsilon\to 0} M\theta(t) = \theta^*\ in\ L^\infty(0,T;L^2(\Omega)) - weak, \\ \lim_{\varepsilon\to 0} Mu(t) = v^*\ in\ L^2(0,T;H^1(\Omega)) - weak, \\ \lim_{\varepsilon\to 0} Mu(t) = v^*\ in\ L^2(0,T;L^2(\Omega)). \end{cases} \tag{4.1}$$

Proof. By Lemma 2.2 and Proposition 3.1, we have

$$\varepsilon\|M\theta\|^2_{L^2(\Omega)} = \|M\theta\|^2_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon(\|M\theta_0\|^2_{L^2(\Omega)} + \|\nabla N\theta_0\|^2_{L^2(\Omega_\varepsilon)}), \tag{4.2}$$

and

$$\varepsilon(\|Mu\|^2_{L^2(\Omega)} + \int_0^T \|\nabla' Mu\|^2_{L^2(\Omega)}) \lesssim \varepsilon(\|M\theta_0\|^2_{L^2(\Omega)} + \|\nabla N\theta_0\|^2_{L^2(\Omega_\varepsilon)} + \|Mu_0\|^2_{L^2(\Omega)} + \|\nabla Nu_0\|^2_{L^2(\Omega_\varepsilon)}). \tag{4.3}$$

Similar to the proof of (2.17) in Proposition 2.2, we also have

$$\varepsilon\|Mu_t\|^2_{L^2(0,T;L^2(\Omega))} \le \varepsilon C(T, \|M\theta_0\|^2_{L^2(\Omega)}, \|\nabla N\theta_0\|^2_{L^2(\Omega_\varepsilon)}, \|Mu_0\|^2_{H^1(\Omega)}, \|\nabla Nu_0\|^2_{H^1(\Omega_\varepsilon)}). \tag{4.4}$$

Therefore, from (4.2)-(4.4), we get the results immediately. □

Now, let's prove Theorem 1.4.

Proof. Due to Proposition 4.1, to justify Theorem 1.4, it is sufficient to verify that (v^*, θ^*) is a pair of weak solution to the 2D system (1.11) and satisfies the condition (1.10).

First, we shall prove (v^*, θ^*) is the weak solution of system (1.11). Let

$$D_1 = \{f|f(x') \in (H^2(\Omega))^2, \nabla' \cdot f = 0, (\nabla' \times f) \times n|_{\partial\Omega} = f \cdot n|_{\partial\Omega} = 0\}, \tag{4.5}$$

and

$$D_2 = \{g|g(x') \in H^1(\Omega)\}. \tag{4.6}$$

Recalling Lemma 2.1, for $\forall (f, g) \in D_1 \times D_2$, applying M to the system (1.1) and then testing f and g on $(1.1)_1$ and $(1.1)_2$ respectively, we get

$$\begin{aligned} \frac{d}{dt}\int_\Omega Mu \cdot f dx' + \nu \int_\Omega \nabla' Mu \cdot \nabla' f dx' \\ + \int_\Omega (Mu \cdot \nabla')Mu \cdot f dx' + \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} (Nu \cdot \nabla)Nu \cdot f dx = 0, \end{aligned} \tag{4.7}$$

and

$$\frac{d}{dt}\int_\Omega M\theta \cdot g dx' + \int_\Omega (Mu \cdot \nabla')M\theta \cdot g dx' + \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} (Nu \cdot \nabla)N\theta \cdot g dx = 0. \tag{4.8}$$

By $H\ddot{o}lder's$ inequality and Proposition 3.1, we have

$$\begin{aligned} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} (Nu \cdot \nabla)Nu \cdot f dx &\le C\varepsilon^{-1}\|Nu\|_{L^3(\Omega_\varepsilon)}\|\nabla Nu\|_{L^2(\Omega_\varepsilon)}\|f\|_{L^6(\Omega_\varepsilon)} \\ &\le C\varepsilon^{-1}\varepsilon^{\frac{1}{2}-\frac{1}{6}}\|\nabla Nu\|^2_{L^2(\Omega_\varepsilon)}\|f\|_{H^1(\Omega)} \\ &\le C\varepsilon^{-1}\varepsilon^{\frac{1}{2}-\frac{1}{6}}\varepsilon^2 R_0^2(1+t)\|f\|_{H^1(\Omega)} \to 0,\ (\varepsilon \to 0), \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} (Nu \cdot \nabla)N\theta M g dx &\le C\varepsilon^{-1}\|Nu\|_{L^3(\Omega_\varepsilon)}\|\nabla N\theta\|_{L^2(\Omega_\varepsilon)}\|g\|_{L^6(\Omega_\varepsilon)} \\ &\le C\varepsilon^{-1}\varepsilon^{\frac{1}{2}-\frac{1}{6}}\|\nabla Nu\|_{L^2(\Omega_\varepsilon)}\|\nabla N\theta\|_{L^2(\Omega_\varepsilon)}\|g\|_{H^1(\Omega)} \\ &\le C\varepsilon^{-1}\varepsilon^{\frac{1}{2}-\frac{1}{6}}\varepsilon R_0(1+t)^{\frac{1}{2}}\|\nabla\theta_0\|_{L^2(\Omega_\varepsilon)}\|g\|_{H^1(\Omega)} \to 0,\ (\varepsilon \to 0). \end{aligned} \tag{4.10}$$

The last inequality in (4.10), we have used the following fact

$$\|\nabla N\theta\|^2_{L^2(\Omega_\varepsilon)} \le \|\theta\|^2_{H^1(\Omega_\varepsilon)} \le C(T, \|\theta_0\|^2_{H^1(\Omega_\varepsilon)}, \|u_0\|^2_{H^2(\Omega_\varepsilon)}). \tag{4.11}$$

which is guaranteed by (2.23).

Recalling Proposition 4.1, and let $\varepsilon \to 0$ in (4.7) and (4.8), we obtain for $\forall (f, g) \in D_1 \times D_2$, there holds

$$\frac{d}{dt}\int_\Omega v^* \cdot f dx' + \nu \int_\Omega \nabla v^* \cdot \nabla f dx' + \int_\Omega (v^* \cdot \nabla)v^* \cdot f dx' = 0, \tag{4.12}$$

and

$$\frac{d}{dt}\int_\Omega \theta^* \cdot g dx' + \int_\Omega (v^* \cdot \nabla')\theta^* g dx' = 0, \tag{4.13}$$

which implies that (v^*, θ^*) is a pair of weak solution to system (1.11) with initial data $(\tilde{v}_0, \tilde{\theta}_0)$.

By setting $f = Mu$ in (4.5), $g = M\theta$ in (4.6) and integrating from 0 to t, we have

$$\int_\Omega |Mu|^2 dx' + \nu \int_0^t \int_\Omega |\nabla' Mu|^2 dx' ds + \frac{1}{\varepsilon} \int_t \int_{\Omega_\varepsilon} (Nu \cdot \nabla) Nu \cdot Mu dx ds = \int_\Omega |Mu_0|^2 dx', \tag{4.14}$$

and

$$\int_\Omega |M\theta|^2 dx' + \frac{1}{\varepsilon} \int_0^t \int_{\Omega_\varepsilon} (Nu \cdot \nabla) N\theta' M\theta dx ds = \int_\Omega |M\theta_0|^2 dx'. \tag{4.15}$$

Similarly to (4.7), we obtain

$$\lim_{\varepsilon \to 0} (\int_\Omega |Mu|^2 dx' + \nu \int_0^t \int_\Omega |\nabla' Mu|^2 dx' ds) = \int_\Omega |\tilde{v}_0|^2 dx', \tag{4.16}$$

and

$$\lim_{\varepsilon \to 0} (\int_\Omega |M\theta|^2 dx') = \int_\Omega |\tilde{\theta}_0|^2 dx'. \tag{4.17}$$

Then by (4.12), (4.13), (4.16) and (4.17), we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} (\int_\Omega |Mu - v^*|^2 dx' + \nu \int_0^t \int_\Omega |\nabla'(Mu - v^*)|^2 dx' ds) \\ &= \lim_{\varepsilon \to 0} (\int_\Omega |Mu|^2 dx' + \nu \int_0^t \int_\Omega |\nabla' Mu|^2 dx' ds) \\ &\quad + (\int_\Omega |v^*|^2 dx' + \nu \int_0^t \int_\Omega |\nabla' v^*|^2 dx' ds) \\ &\quad - \lim_{\varepsilon \to 0} 2 (\int_\Omega Mu \cdot v^* dx' + \nu \int_0^t \int_\Omega \nabla' Mu \cdot \nabla' v^* dx' ds) = 0, \end{aligned} \tag{4.18}$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega |M\theta - \theta^*|^2 dx' = \lim_{\varepsilon \to 0} \int_\Omega |M\theta|^2 dx' - \lim_{\varepsilon \to 0} 2 \int_\Omega M\theta \cdot \theta^* dx' + \int_\Omega |\theta^*|^2 dx' = 0, \tag{4.19}$$

which completes Theorem 1.4.

□

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