THE REGULARITY OF SOLUTION FOR A GENERALIZED HUNTER-SAXTON TYPE EQUATION

HONG CAI, GENG CHEN, AND YANNAN SHEN

Abstract. The cusp singularity, with only Hölder continuity, is a typical singularity formed in the quasilinear hyperbolic partial differential equations, such as the Hunter-Saxton and Camassa-Holm equations. We establish the global existence of Hölder continuous energy conservative weak solution for a family of Hunter-Saxton type equations, where the regularity of solution varies with respect to a parameter. This result can help us predict regularity of cusp singularity for many other models.

Keywords. Nonlinear wave equations, regularity, cusp singularity.

1. Introduction

We consider a family of nonlinear wave equations parameterized by λ, which have the form

$$u_{tx} + f'(u)\, u_{xx} + \lambda\, f''(u)\, u_x^2 = g(u, u_x)\,. \tag{1.1}$$

Here $u = u(x,t)$ is a scalar function defined for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. More intuitively, equation (1.1) can be formally written as

$$u_{tx} + (f'(u))^{1-\lambda} \left[(f'(u))^{\lambda}\, u_x\right]_x = g(u, u_x)\,.$$

Equation (1.1) includes several important and interesting models when λ takes different values.

- When $\lambda = 1,\ g(u) = 0$: $u_t + (f(u))_x = 0$ is a scalar hyperbolic conservation law. The solution in general forms discontinuities (shock waves). We study BV existence. See [12].
- When $\lambda = \frac{1}{2},\ f'(u) = u,\ g(u) = 0$: $u_{tx} + u\, u_{xx} + \frac{1}{2}(u_x)^2 = 0$ is the Hunter-Saxton equation, a simplified model from nematic liquid crystals [2, 4, 13, 14, 15], where global H^1 Solutions were found [4, 5, 8, 13, 14, 15], after the formation of cusp singularity.
- When $\lambda = \frac{1}{2a},\ f'(u) = u^a$ with any positive integer a and $g(u, u_x) = u^{a+1} - P - Q_x$:

$$u_{tx} + u^a u_{xx} + \tfrac{1}{2} u^{a-1} (u_x)^2 - u^{a+1} + P + Q_x = 0. \tag{1.2}$$

  where $P := p * [\frac{2a-1}{2} u^{a-1}(u_x)^2 + u^{a+1}]$ and $Q := \frac{(a-1)}{2}\, p * [u^{a-2}(u_x)^3]$. This equation gives Camassa-Holm equation when $a = 1$ ($\lambda = \frac{1}{2}$) with H^1 solution [1, 3], and Novikov equation when $a = 2$ ($\lambda = \frac{1}{4}$) with $W^{1,4}$ solution [6, 7].

One common feature of these equations is the finite time gradient blowup of solutions even with smooth initial data. The regularity of (1.1), has been considered by two earlier papers of the authors: in [10] when $g = 0$, we showed that there exists a global energy conservative Hölder continuous weak solution for (1.1) with exponent $1 - \lambda$, when $\lambda \in (0, \frac{1}{2})$. Later in [11], we proved that for the global existence of conservative type Hölder continuous solutions for (1.2), i.e. (1.1) with non-local source g, with exponent $1 - \lambda = 1 - \frac{1}{2a}$, for $a \geq 1$, i.e. when $\lambda \leq \frac{1}{2}$. Here the regularity of solution is especially consistent with earlier results on Camassa-Holm and Novikov equations.

Now we propose the following conjecture.

Conjecture: *When $\lambda \in (0,1)$, for the initial value problem of (1.1) with reasonable condition on g, there is a global Hölder continuous solution on both x and t with exponent $1 - \lambda$.*

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Loosely speaking, the Hölder exponent $1-\lambda$ in the conjecture comes from the one dimensional Sobolev embedding $W^{1,\frac{1}{\lambda}} \hookrightarrow C^{0,1-\lambda}$, where the equation (1.1) satisfy energy conservation or balance law on $(u_x)^{1/\lambda}$. The regularity of Hölder continuous solution including cusp singularity in the conjecture is basically consistent with existing results at the ending values when $\lambda = 0$ and 1.

In this paper, we address one big missing piece for this conjecture: the regularity of solution when $\gamma \in (\frac{1}{2}, 1)$. For simplicity, we only consider the case when the source term $g = 0$.

Theorem 1. *We consider the initial boundary value problem of* (1.1) *on the region* $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ *with constant parameter* $\lambda \in (0,1)$ *and* $g = 0$*. As initial and boundary data we take*

$$u(0,t) = 0, \qquad u(x,0) =: u_0(x) \in W^{1,\frac{1}{\lambda}}_{loc}(\mathbb{R}^+)\,, \tag{1.3}$$

and a compatibility condition

$$u_0(0) = 0 \quad \textit{and} \quad u_0'(0) = 0\,. \tag{1.4}$$

We further assume that $f(u)$ *is a* C^2 *function satisfying*

$$f'(0) \geq 0, \qquad |f''(u_1) - f''(u_2)| \leq L|u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R}$$

for some constant L*, and* $|f''(u)|$ *is uniformly bounded.*

For any given time $T \in (0,\infty)$*, this initial-boundary value problem admits a weak solution* $u(x,t)$ *defined on* $\mathbb{R}^+ \times [0,T]$ *in the following sense.*

(i). *The equation (1.1) is satisfied in the weak sense, i.e., for any* $\phi \in C^1_c(\mathbb{R}^+ \times [0,T])$,

$$\int_0^T \int_0^\infty \Big\{ -u_x\,(\phi_t + f'(u)\,\phi_x) + (\lambda - 1)\,f''(u)u_x^2\,\phi \Big\}\,dx\,dt = 0. \tag{1.5}$$

(ii). *The initial and boundary conditions* (1.3)–(1.4) *are satisfied point-wise.*

(iii). *For any fixed* $t > 0$*, the function* $u(\cdot,t)$ *is in* $W^{1,\frac{1}{\lambda}}_{loc}(\mathbb{R}^+)$*, hence is locally Hölder continuous with exponent* $1-\lambda$ *by the Sobolev embedding Theorem.*

Here $W^{1,\frac{1}{\lambda}}_{loc}(\mathbb{R}^+)$ is the Sobolev space with standard notation. The assumption that $f'(0) \geq 0$ protects that the wave on the boundary $x = 0$ does not flow in an outward direction.

The rest of this paper is organized as follows. In Section 2, we will introduce the characteristic coordinates and semilinear equations. In section 3, we show the existence of semilinear equations, then in section 4, we do the reverse transformation to obtain solutions for the original equation.

2. New coordinates

In this section, we derive a semi-linear system of (1.1) for smooth solutions by introducing a set of new variables. Consider the equation of the characteristic

$$\frac{dx^c(t)}{dt} = f'\big(u(x^c(t),t)\big).$$

The characteristic passing through the point (x,t) will be denoted by

$$a \mapsto x^c(a;\, x,\, t) \quad \text{or equivalently} \quad b \mapsto t^c(b;\, x,\, t),$$

where a and b are the time and space variables of the characteristic, respectively. Then we introduce the change of coordinates $(x,t) \mapsto (Y,\tau)$ where

$$Y := \begin{cases} \int_0^{x^c(0;\,x,\,t)} \big(1 + u_x^2(x',0)\big)^{\frac{1}{2\lambda}}\,dx', \\ \qquad \text{when the characteristic passing } (x,t) \text{ interacts } t = 0; \\ \\ -t^c(0;\,x,\,t)\,f'(0) \\ \qquad \text{when the characteristic passing } (x,t) \text{ interacts } x = 0, \end{cases} \tag{2.1}$$

with $(x,t)\in\mathbb{R}^+\times[0,T]$ and $\tau := t$. This implies

$$Y_t + f'(u)Y_x = 0, \quad \tau_t = 1 \quad \text{and} \quad \tau_x = 0,$$

from which, one has for any smooth function m,

$$\begin{cases} m_t + f'(u)m_x = m_Y\,(Y_t + f'(u)Y_x) + m_\tau\,(\tau_t + f'(u)\tau_x) = m_\tau, \\ m_x = m_Y\,Y_x + m_\tau\,\tau_x = m_Y\,Y_x. \end{cases} \tag{2.2}$$

Now we formulate a set of equations on (Y,τ)-coordinates which is equivalent to (1.1). The following further variables will be used:

$$v := 2\arctan u_x \quad \text{and} \quad \xi := \frac{(1+u_x^2)^{\frac{1}{2\lambda}}}{Y_x}.$$

By a direct calculation, we obtain a semi-linear system, c.f. [10]

$$\begin{cases} u_Y &= \xi\,\sin\frac{v}{2}\,(\cos\frac{v}{2})^{\frac{1}{\lambda}-1}\,, \\ v_\tau &= -2\lambda\,f''(u)\sin^2\frac{v}{2}\,, \\ \xi_\tau &= \frac{1}{2}\,\xi\,f''(u)\,\sin v\,. \end{cases} \tag{2.3}$$

We remark here that semi-linear system (2.3) is invariant under translation by 2π in v. It would be more precise to use e^{iv} as variable. For simplicity, we use $v\in[-\pi,\pi]$ with endpoints identified.

We now consider the initial-boundary conditions of system (2.3) in the coordinates (Y,τ), corresponding to (1.3) and (1.4) in the original coordinates (x,t). By (2.1), we know that, the initial lines $\tau=0$ and $t=0$ are the same line. Moreover, the line $x=0$ $(t\geq 0)$ in the (x,t) plane is transformed to a curve $Y:=\Gamma_b(\tau)$ in the (Y,τ) plane defined by

$$Y = \Gamma_b(\tau) = -f'(0)\tau\,.$$

Recall $f'(0)\geq 0$. Thus, the coordinates transformation maps the domain $(\mathbb{R}^+,[0,T])$ in the (x,t) plane into the set

$$\Omega := \{(Y,\tau);\ Y\geq\Gamma_b(\tau),\ 0\leq\tau\leq T\}$$

in the (Y,τ) plane. Then we can supplement the initial-boundary data for (2.3) as follows.

The initial data on $(Y,0)$ with $Y\geq 0$ are

$$\begin{cases} u(Y,0) &:= u_0\big(x(Y,0)\big), \\ v(Y,0) &:= 2\arctan(u_0'\big(x(Y,0)\big)), \\ \xi(Y,0) &:= 1. \end{cases} \tag{2.4}$$

The boundary conditions on $Y=\Gamma_b(\tau)$ with $0\leq\tau\leq T$ are

$$\begin{cases} u(\Gamma_b(\tau),\tau) &:= 0, \\ v(\Gamma_b(\tau),\tau) &:= 0, \\ \xi(\Gamma_b(\tau),\tau) &:= 1. \end{cases} \tag{2.5}$$

3. Existence on the new coordinates

Now we prove the existence of solution for system (2.3) with initial and boundary data (2.4)–(2.5).

Theorem 2. *Assume all conditions on initial and boundary data in Theorem 1 hold. Then the corresponding problem (2.3) with initial-boundary data (2.4)–(2.5) has a solution defined for all $(Y,\tau)\in\Omega$.*

Proof. We first show a local existence result by finding a local solution as fixed point of a suitable integral transformation. To this end, we divide the region Ω into two subregions: $\Omega=\Omega_1\cup\Omega_2$, with

$$\Omega_1 := \{(Y,\tau);\ \tau\geq 0,\ \Gamma_b(\tau)\leq Y\leq 0\}, \qquad \Omega_2 := \{(Y,\tau);\ \tau\geq 0,\ Y\geq 0\},$$

Specifically, if $f'(0)=0$, then $\Omega=\Omega_2$.

Now we first discuss a local existence of solutions in the region Ω_1 near the point $(0,0)$. For fixed constant $K_1>0$ and for some $0<\alpha\le 1-\lambda$, consider a set

$$\mathcal{K}_1=\Big\{(u,v,\xi)\Big|\|(u,v,\xi)(Y,\tau)\|_{C^\alpha(\Omega_{1\delta_1})}\le K_1,\ \|(u_Y,v_\tau,\xi_\tau)(Y,\tau)\|_{L^\infty(\Omega_{1\delta_1})}\le K_1,\\ (u,v,\xi)(\Gamma_b(\tau),\tau)=(0,0,1)\Big\},$$

where $\delta_1>0$ is a small constant and

$$\Omega_{1\delta_1}=\big\{(Y,\tau)\in\Omega_1:\ dist((Y,\tau),(0,0))\le\delta_1\big\}.$$

For $(u,v,\xi)\in\mathcal{K}_1$, consider a map

$$\mathcal{T}_1(u,v,\xi)=(\hat u,\hat v,\hat\xi)$$

defined by the solution of (2.3). More precisely,

$$\hat u(Y,\tau)=\int_{\Gamma_b(\tau)}^{Y}\xi\,\sin\frac{v}{2}\,(\cos\frac{v}{2})^{\frac{1}{\lambda}-1}(Y',\tau)\,dY',\tag{3.1}$$

$$\hat v(Y,\tau)=-2\lambda\int_{\Gamma_b^{-1}(Y)}^{\tau}f''(u)\sin^2\frac{v}{2}(Y,\tau')\,d\tau',\tag{3.2}$$

$$\hat\xi(Y,\tau)=1+\frac{1}{2}\int_{\Gamma_b^{-1}(Y)}^{\tau}\xi f''(u)\,\sin v\,(Y,\tau')\,d\tau'.\tag{3.3}$$

It is clear that $(\hat u,\hat v,\hat\xi)(\Gamma_b(\tau),\tau)=(0,0,1)$ and the space $\mathcal{K}_1$ is compact in $C^0(\Omega_{1\delta_1})$. From the Schauder fixed point theorem it follows that to obtain the existence of solutions of system (2.3) in Ω_1, our goal now is to show that the map $\mathcal{T}_1$ is continuous under C^0 norm and maps $\mathcal{K}_1$ to itself. In fact, a straightforward computation can check that the map $\mathcal{T}_1$ is continuous. By (3.1)–(3.3), if δ_1 is small enough, $\mathcal{T}_1$ maps $\mathcal{K}_1$ to itself. For example, $\hat u_Y$ is bounded, so $\hat u$ is Lipschitz in the Y-direction. In the τ direction, the C^α norm of $\hat u$ is less than or equal to $C\delta_1$ times $\|(u,v,\xi)(Y,\tau)\|_{C^\alpha}$ for some constant C. We refer readers to [9] for more details.

Therefore, we have a fixed point $(u,v,\xi)\in\mathcal{K}_1$ for the map $\mathcal{T}_1$.

Similarly, for the local existence in the region Ω_2 near the curve $\tau=0$, we apply the fixed point argument to the following maps obtained by integrating the system (2.3):

$$\mathcal{T}_2(u,v,\xi)=(\bar u,\bar v,\bar\xi),$$

where

$$\bar u(Y,\tau)=u(0,\tau)+\int_0^{Y}\xi\,\sin\frac{v}{2}\,(\cos\frac{v}{2})^{\frac{1}{\lambda}-1}(Y',\tau)\,dY',$$

$$\bar v(Y,\tau)=2\arctan(u_0'(x(Y,0)))-2\lambda\int_0^{\tau}f''(u)\sin^2\frac{v}{2}(Y,\tau')\,d\tau',$$

$$\bar\xi(Y,\tau)=1+\frac{1}{2}\int_0^{\tau}\xi\,f''(u)\,\sin v\,(Y,\tau')\,d\tau',$$

for any point $(Y,\tau)\in\Omega_{2\delta_1}:=\Omega_2\cap\{\tau<\delta_1\}$. For sufficiently large constant $K_2>0$, define the set

$$\mathcal{K}_2=\Big\{(u,v,\xi)\Big|\|(u,v,\xi)(Y,\tau)\|_{C^\alpha(\Omega_{2\delta_2})}\le K_2,\ \|(u_Y,v_\tau,\xi_\tau)(Y,\tau)\|_{L^\infty(\Omega_{2\delta_2})}\le K_2,\\ (u,v,\xi)(0,\tau)=(\widetilde u,\widetilde v,\widetilde\xi)(\tau)\Big\},$$

where $\delta_2\le\delta_1$ is a small constant and

$$\Omega_{2\delta_2}=\big\{(Y,\tau)\in\Omega_{2\delta_1}:\ dist((Y,\tau),\{\tau=0\})\le\delta_2\big\}.$$

Note that the functions $(u,v,\xi)(0,\tau)$ are achieved in $\Omega_{1\delta_1}$ by map $\mathcal{T}_1$, so $(u,v,\xi)(0,\tau)$ are C^α continuous. It is clear that $(\bar u,\bar v,\bar\xi)(0,\tau)=(\widetilde u,\widetilde v,\widetilde\xi)(\tau)$ and the space $\mathcal{K}_2$ is compact in $C^0(\Omega_{2\delta_2})$.

Hence, similar to the existence in $\Omega_{1\delta_1}$, by selecting appropriate constants K_2 and then δ_2, the map $\mathcal{T}_2$ is continuous under C^0 norm and maps $\mathcal{K}_2$ to itself, the local existence of solutions in Ω_2 will follow from the Schauder fixed point theorem.

To extend this local solution to the whole domain Ω, one must establish a priori bounds, showing that ξ remains bounded on bounded sets. This is obviously. Since $f''(u)$ is uniformly bounded above, using the last equation in (2.3), we could find a priori upper bound of ξ, i.e.

$$\xi \leq exp\{\frac{1}{2}T \max_{(Y,\tau)\in\Omega}(f''(u))\}.$$

Thus we have the global existence Theorem 2. □

4. Solutions in the original variables

In this section, we show that the solution in the (Y,τ)-coordinate can be expressed by the original variables (x,t). The proof will be given in several steps.

Step 1. Firstly, we need the inverse transformation on the solution of the semi-linear system to construct the solution to (1.1). By (2.2), set $t=\tau$ and

$$x_Y = \xi(\cos^2\frac{v}{2})^{\frac{1}{2\lambda}}, \qquad x_\tau = f'(u). \tag{4.1}$$

By a direct calculation, one has $x_{\tau Y} = f''(u)\xi\,\sin\frac{v}{2}\,(\cos\frac{v}{2})^{\frac{1}{\lambda}-1} = x_{Y\tau}$, so two equations in (4.1) are equivalent. Hence, if the solution exists, we can define the function $x(Y,\tau)$ based on the region where (Y,τ) is located. For example, if $(Y,\tau)\in\Omega_1$, then

$$x(Y,\tau) = \int_{\Gamma_b(\tau)}^{Y} \xi(\cos^2\frac{v}{2})^{\frac{1}{2\lambda}}(Y',\tau)\,dY' = \int_{\Gamma_b^{-1}(Y)}^{\tau} f'(u)(Y,\tau')\,d\tau'.$$

If $(Y,\tau)\in\Omega_2$, then

$$x(Y,\tau) = \int_{\Gamma_b(\tau)}^{Y} \xi(\cos^2\frac{v}{2})^{\frac{1}{2\lambda}}(Y',\tau)\,dY' = \int_0^Y \xi(\cos^2\frac{v}{2})^{\frac{1}{2\lambda}}(Y',0)\,dY' + \int_0^\tau f'(u)(Y,\tau')\,d\tau'.$$

Step 2. We claim that a solution of (1.1) can be obtained by setting

$$u(x,t) = u\big(x(Y,\tau),t(\tau)\big)$$

although the map $(Y,\tau)\to(x,t)$ constructed above may not be one-to-one. That is, for any fixed point (x,t), the values of u do not depend on the choice of Y. In fact, if $x(Y_1,\tau^*) = x(Y_2,\tau^*) = x^*$, for two points (Y_1,τ^*) and (Y_2,τ^*) in Ω and $Y_1<Y_2$, then by the monotonicity of x on Y given in (4.1), we obtain $x_Y(Y,\tau^*) = \xi(\cos\frac{v}{2})^{\frac{1}{\lambda}} = 0$ for $Y\in[Y_1,Y_2]$. This yields $\cos\frac{v}{2}=0$ for $Y\in[Y_1,Y_2]$. Moreover, by (2.3), then $u_Y=0$ when $Y\in[Y_1,Y_2]$. Therefore, we get $u(Y_1,\tau^*) = u(Y_2,\tau^*)$.

Step 3. Finally, the function $u(x,t)$ is in the space $W_{loc}^{1,\frac{1}{\lambda}}(\mathbb{R}^+)$. For any time $t\in[0,T]$ and bounded interval $[x_1,x_2]\in\mathbb{R}^+$, recalling (2.3) and (4.1), one gets

$$\int_{x_1}^{x_2} |u_x|^{\frac{1}{\lambda}}\,dx = \int_{Y_1}^{Y_2}\Big|\frac{u_Y}{\xi(\cos\frac{v}{2})^{\frac{1}{\lambda}}}\Big|^{\frac{1}{\lambda}}\xi(\cos\frac{v}{2})^{\frac{1}{\lambda}}\,dY = \int_{Y_1}^{Y_2}|\sin\frac{v}{2}|^{\frac{1}{\lambda}}\xi\,dY < \infty.$$

Hence, the function $u(\cdot,t)\in W_{loc}^{1,\frac{1}{\lambda}}(\mathbb{R}^+)$ for any $t>0$. By the Sobolev embedding Theorem, this implies the locally Hölder continuity with exponent $1-\lambda$ of u as a function of x.

Step 4. Finally, we prove that the function $u = u(x,t)$ provides a weak solution of (1.1). For any test function $\phi \in C_c^1(\mathbb{R}^+ \times [0,T])$, by (2.3) and (4.1), we see that

$$\begin{aligned}
&\int_0^T \int_0^\infty \Big\{ -u_x\,(\phi_t + f'(u)\,\phi_x) + (\lambda-1)f''(u)u_x^2\,\phi \Big\}\,dx\,dt \\
&= \int\int_{(Y,\tau)\in\Omega} \Big\{ -u_Y\,\phi_\tau + (\lambda-1)f''(u)\xi\sin^2\tfrac{v}{2}(\cos\tfrac{v}{2})^{\frac{1}{\lambda}-2}\,\phi \Big\}\,dY\,d\tau \\
&= \int\int_{(Y,\tau)\in\Omega} \Big\{ (\xi\,\sin\tfrac{v}{2}\,(\cos\tfrac{v}{2})^{\frac{1}{\lambda}-1})_\tau + (\lambda-1)f''(u)\xi\sin^2\tfrac{v}{2}(\cos\tfrac{v}{2})^{\frac{1}{\lambda}-2} \Big\}\phi\,dY\,d\tau \\
&= 0,
\end{aligned}$$

which is exactly (1.5). This completes the proof of Theorem 1.

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Hong Cai
School of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, 266061, China.
Email address: caihong19890418@163.com

Geng Chen
Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA.
Email address: gengchen@ku.edu

Yannan Shen
Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA.
Email address: yshen@ku.edu